On manifolds admitting the consistent Lagrangian formulation for higher spin fields

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Abstract

We study a possibility of Lagrangian formulation for free higher spin bosonic totally symmetric tensor field on the background manifold characterizing by the arbitrary metric, vector and third rank tensor fields in framework of BRST approach. Assuming existence of massless and flat limits in the Lagrangian and using the most general form of the operators of constraints we show that the algebra generated by these operators will be closed only for constant curvature space with no nontrivial coupling to the third rank tensor and the strength of the vector fields. This result finally proves that the consistent Lagrangian formulation at the conditions under consideration is possible only in constant curvature Riemann space.

Lagrangian formulation of interacting higher spin field theory is a fundamental unsolved problem of classical field theory during long time (see e.g. the reviews [1]). The essence of the problem is that any naive (e.g. minimal) including the interaction to free higher spin field Lagrangian violates consistency of the equations of motion (see the various aspects of the inconsistency in [2, 3, 4]). One of the partial aspects of the generic problem is a Lagrangian formulation for higher spin fields coupled to external background. At present, all known consistent Lagrangian formulations are constructed only on space of constant curvature without any other external fields. Then a natural question arises if there exist the other background

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fields admitting the consistent Lagrangian formulation for higher spin fields. For example, if to accept that the massive higher spin fields have the superstring origin, the evident background in bosonic sector is formed by the fields from massless string spectrum what corresponds in sigma-model approach [5] to manifold endowed with Riemann metric and additional external scalar (dilaton), vector and totally antisymmetric third rank tensor field which can be associated with torsion. In principle one can hope that the consistent Lagrangian formulation of higher spin fields coupled to background fields actually exists under some equations linking all the background fields.

In this note we consider the free massive higher spin bosonic field theory coupled to external metric, vector field and arbitrary third rank tensor field and assume that the Lagrangian contains no inverse mass terms. At these conditions we prove that the only manifold admitting the consistent Lagrangian formulation is constant curvature space with vanishing scalar, vector and third rank tensor external fields. The proof is based on generic BRST formulation of higher spin field theory (see the various use of BRST formalism in higher spin field theory in [6, 7, 8, 9, 10, 11] which allows efficiently to study the bosonic and fermionic, massless and massive higher spin fields, to take into account a gauge structure of the theory and to work with tensor fields of various symmetry of indices.

The BRST approach to Lagrangian construction for higher spin fields, which is developed in our papers, is realized as follows. The mass-shell equation and subsidiary conditions are treated as a part of the first class constraints of some unknown yet gauge theory. The new constraints are added to initial ones to form a complete set of first class constraints. All the constraints are formulated as the operators acting in auxiliary Fock space and it is assumed that the algebra of the constraints in terms of commutators is closed. Taking into account these constraints one can construct the Hermitian nilpotent BRST-BFV operator Q [12] and find higher spin field Lagrangian in terms of the Q. Nilpotency of the Q guarantees a gauge invariance of the corresponding action. As a result we get the higher spin field Lagrangian which automatically contains all the auxiliary fields. Such an approach has been completely realized in d-dimensional flat and AdS spaces and its different aspects have been studied in [6, 7, 8, 10]. The essential basic element of such approach was closure of the algebra of operator constraints.

To realize this approach for the fields on some background first of all we should find the corresponding mass-shell and subsidiary conditions. For bosonic fields in flat space they are Klein-Gordon equation and the conditions for the fields to be divergence free and traceless (see e.g. [13])

$$(\partial^2 - m^2)\varphi_{\mu_1\ldots\mu_s} = 0, \qquad \partial^{\mu_1}\varphi_{\mu_1\ldots\mu_s} = 0, \qquad \eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\ldots\mu_s} = 0. \tag{1}$$

In case of fields on a background all the conditions (1) should be deformed by the proper way. For the bosonic fields in AdS space such deformations has been constructed in [8]. It is interesting to point out that in the case of fields in AdS space the corresponding closed algebra of constraints belongs to class of quadratic non-linear algebra and finding the nilpotent BRST operator is a very nontrivial problem (see e.g. [14]). In the case under consideration we should construct a deformation of the constraints (1), realizing all the conditions as the operator constraints and find the restrictions on the background when their algebra will be closed.

We assume that a manifold under consideration is endowed with metric $g_{\mu\nu}$, the background vector A_μ and third rank tensor $K_{\mu\nu\alpha}$ field with no index symmetry, so that the torsion tensor

is a particular case of $K_{\mu\nu\alpha}$ tensor. The covariant derivatives are constructed with the use of Christoffel symbols $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$.

Introduce the auxiliary Fock space generated by bosonic creation and annihilation operators with tangent space indices $(a, b = 0, 1, \ldots, d-1)$

$$[a_a, a_b^+] = \eta_{ab}, \qquad \eta_{ab} = diag(-, +, \ldots, +). \tag{2}$$

An arbitrary vector in this Fock space has the form

$$|\varphi\rangle \;=\; \sum_{s=0}^{\infty} \varphi_{a_1\ldots a_s}(x)\, a^{+a_1} \ldots a^{+a_s}|0\rangle = \sum_{s=0}^{\infty} \varphi_{\mu_1\ldots\mu_s}(x)\, a^{+\mu_1} \ldots a^{+\mu_s}|0\rangle \equiv \sum_{s=0}^{\infty} |\varphi_s\rangle, \tag{3}$$

where $a^{+\mu}(x) = e^\mu_a(x)a^{+a}$, $a^\mu(x) = e^\mu_a(x)a^a$, with $e^\mu_a(x)$ being the vielbein. It is evident that $[a_\mu, a_\nu^+] = g_{\mu\nu}$. We also suppose the standard relation $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\alpha{}_{\nu\mu} e^a_\alpha + \omega_\mu{}^a{}_b e^b_\nu = 0$, where $\omega_\mu{}^a{}_b$ is the spin connection.

Then one introduces derivative operator

$$D_\mu = \partial_\mu + \omega^{ab}_\mu a_a^+ a_b, \qquad\qquad D_\mu|0\rangle = 0 \tag{4}$$

which acts on states of the Fock space (3) as the covariant derivative

$$D_\mu|\varphi_s\rangle = (\nabla_\mu \varphi_{\mu_1\ldots\mu_s})\, a^{+\mu_1} \ldots a^{+\mu_s}|0\rangle \tag{5}$$

and tries to realize the generalization of equations (1) in the operator form

$$l_0|\varphi_s\rangle = l_1|\varphi_s\rangle = l_2|\varphi_s\rangle = 0 \tag{6}$$

where operators l_0, l_1, l_2 corresponding to Klein-Gordon, divergence free and traceless equations respectively.

The procedure of Lagrangian construction based on the BRST method looks as follows. For the Lagrangian be a real function the BRST operator used for its construction must be a Hermitian operator. It assumes that the set of operators underlying the BRST operator must be invariant under Hermitian conjugation. To have such a set of operators we add to constraints l_0, l_1, l_2 their Hermitian conjugated operators l_1^+, l_2^+ with l_0 being assumed to be self-conjugated. Then for constructing the BRST operator the underlying set of operators must form an algebra. To get the algebra we must add to operators l_0, l_1, l_2, l_1^+, l_2^+ some more operators providing closing the algebra. But if we want to construct with the help of the obtained algebra Lagrangian for spin-s field, then this algebra must be a deformation of the algebra in Minkowski or in AdS [8] space. Thus we can add only two operators which are generalization of operators

$$g_0 = a_\mu^+ a^\mu + \tfrac{d}{2}, \qquad g_m = m^2 + const \tag{7}$$

to the case of curved space. Since operator g_0 is dimensionless and we do not consider terms with inverse powers of the mass then it is impossible to deform operator g_0 by terms with the curvature or with the background fields.[1] Therefore operator g_0 keeps the same form (7) as in the flat case. As for possible generalization of operator g_m we postpone this question. Thus we

[1] We adopt that the backgound fields have mass dimension one like the torsion.

came to the conclusion that in order to be possible to construct Lagrangian with the help of the BRST method we must find explicit expressions for the operators

$$l_0, \qquad l_1, \qquad l_1^+, \qquad l_2, \qquad l_2^+, \qquad g_0, \qquad g_m \tag{8}$$

so that they form an algebra.[2] If we find algebra different from the AdS case, then it means that there is a hope to construct Lagrangian in space different from AdS.

We will deform the operators by introducing background fields. We explore the case when the background fields are vector and third rank tensor with dimension of mass. Note that any third rank tensor can be decomposed into totally symmetric $S_{(\mu\nu\sigma)}$, totally antisymmetric $A_{[\mu\nu\sigma]}$ tensors, and tensors with mixed symmetry of the indices (see e.g. [15]). We consider the case of the decomposition when the mixed symmetry tensors have the following symmetry of the indices

$$M_{\mu\nu\sigma} = -M_{[\nu\mu]\sigma}\,, \qquad M_{[\mu\nu\sigma]} = 0\,. \tag{9}$$

In addition we adopt that all the background tensors are traceless, absorbing their traces into vector field V_μ.

Let us discuss possible form of the operators. First, the dimensionless operators

$$l_2 = \tfrac{1}{2}\, a^\mu a_\mu\,, \qquad l_2^+ = \tfrac{1}{2} a_\mu^+ a^{\mu+}\,, \qquad g_0 = a_\mu^+ a^\mu + \tfrac{d}{2} \tag{10}$$

can't be modified by the background fields if we don't take into account the mass in the inverse powers.

Next let us consider operator l_1 responsible for the physical field to be divergence free. Since this operator has mass dimension one then the background fields are introduced linearly in l_1. Also we note that in the terms with the background fields the creation and annihilation operators, if they not contracted with the background fields, are contracted with each other and these contractions can be expressed through the operators l_2, l_2^+, g_0 (10). Therefore the most general expression for the operator l_1 is

$$\begin{aligned} l_1 \;=\;& a^\mu D_\mu + \sum_{k,m=0}^{\infty} \alpha_{km}\, V_\mu a^\mu\, (l_2^+)^m\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \omega_{km}\, V_\mu a^{+\mu}\, (l_2^+)^m\, g_0^k\, l_2^{m+1} \\ &+ \sum_{k,m=0}^{\infty} \beta_{km}\, M_{[\mu\nu]\sigma} a^{+\mu} a^\nu a^\sigma\, (l_2^+)^m\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \sigma_{km}\, M_{[\mu\nu]\sigma} a^{+\nu} a^{+\sigma} a^\mu\, (l_2^+)^m\, g_0^k\, l_2^{m+1} \\ &+ \sum_{k,m=0}^{\infty} \epsilon_{km}\, S_{(\mu\nu\sigma)} a^{+\mu} a^{+\nu} a^{+\sigma}\, (l_2^+)^m\, g_0^k\, l_2^{m+2} + \sum_{k,m=0}^{\infty} \zeta_{km}\, S_{(\mu\nu\sigma)} a^{+\mu} a^{+\nu} a^\sigma\, (l_2^+)^m\, g_0^k\, l_2^{m+1} \\ &+ \sum_{k,m=0}^{\infty} \gamma_{km}\, S_{(\mu\nu\sigma)} a^{+\mu} a^\nu a^\sigma\, (l_2^+)^m\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \theta_{km}\, S_{(\mu\nu\sigma)} a^\mu a^\nu a^\sigma\, (l_2^+)^{m+1}\, g_0^k\, l_2^m \end{aligned} \tag{11}$$

with arbitrary coefficients α_{km}, β_{km}, γ_{km}, ω_{km}, σ_{km}, ϵ_{km}, ζ_{km}, θ_{km}. The totally antisymmetric tensor $A_{[\mu\nu\sigma]}$ cannot be introduced into l_1 since any its contraction with creation and annihilation operators gives zero. But it should be noted that in case we considered the dynamics of a

[2]It should be note that in general case, operators l_0, l_1, l_2 may not coincide with that given in (6). We demand here only that operators (8) have the proper free limit. The consistent conditions on the field (6) will be followed from the BRST construction.

field with mixed symmetry of the indices in the background fields then the totally antisymmetric field $A_{[\mu\nu\sigma]}$ could be introduced into l_1.

Taking Hermitian conjugation (11) and moving operators l_2^+, g_0, l_2 to the right we obtain expression for the operator l_1^+

$$
\begin{aligned}
l_1^+ \;=\; & -a^{+\mu}D_\mu + \sum_{k,m=0}^{\infty} \alpha'_{km}\, V_\mu a^{+\mu}\, (l_2^+)^m\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \omega'_{km}\, V_\mu a^{\mu}\, (l_2^+)^{m+1}\, g_0^k\, l_2^m \\
& + \sum_{k,m=0}^{\infty} \beta'_{km}\, M_{[\mu\nu]\sigma} a^{+\nu}a^{+\sigma}a^{\mu}\, (l_2^+)^m\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \sigma'_{km}\, M_{[\mu\nu]\sigma} a^{+\mu}a^{\nu}a^{\sigma}\, (l_2^+)^{m+1}\, g_0^k\, l_2^m \\
& + \sum_{k,m=0}^{\infty} \epsilon'_{km}\, S_{(\mu\nu\sigma)} a^{\mu}a^{\nu}a^{\sigma}\, (l_2^+)^{m+2}\, g_0^k\, l_2^m + \sum_{k,m=0}^{\infty} \zeta'_{km}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{\nu}a^{\sigma}\, (l_2^+)^{m+1}\, g_0^k\, l_2^m \\
& + \sum_{k,m=0}^{\infty} \gamma'_{km}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{+\nu}a^{\sigma}\, (l_2^+)^m\, g_0^k\, l_2^m \\
& + \sum_{k,m=0}^{\infty} \theta'_{km}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{+\nu}a^{+\sigma}\, (l_2^+)^m\, g_0^k\, l_2^{m+1}
\end{aligned}
\tag{12}
$$

where the primed coefficients can be expressed through non-primed ones and vice versa. Moreover the dependent and independent coefficients can be chosen in a variety of ways, choosing as independent coefficients partially both primed and non-primed coefficients. Note that the terms containing at least one of the operators l_2^+, g_0, l_2 don't influence on closing the algebra (and as a consequence on the background geometry) and this fact we denote as follows

$$
l_1 \;\approx\; a^{\mu}D_\mu + \alpha_{00}\, V_\mu a^{\mu} + \beta_{00}\, M_{[\mu\nu]\sigma} a^{+\mu}a^{\nu}a^{\sigma} + \gamma_{00}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{\nu}a^{\sigma} \tag{13}
$$

$$
l_1^+ \;\approx\; -a^{+\mu}D_\mu + \alpha'_{00}\, V_\mu a^{+\mu} + \beta'_{00}\, M_{[\mu\nu]\sigma} a^{+\nu}a^{+\sigma}a^{\mu} + \gamma'_{00}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{+\nu}a^{\sigma} \tag{14}
$$

where $\approx$ means "up to terms proportional to operators l_2^+, g_0, l_2". Also we note that α_{00}, α'_{00}, β_{00}, β'_{00}, γ_{00}, γ'_{00} can be considered as independent of each other.

Let us consider commutators

$$
[\,l_1, l_2] \approx \gamma_{00}\, S_{(\mu\nu\sigma)} a^{\mu}a^{\nu}a^{\sigma}\,, \qquad\qquad [\,l_1^+, l_2^+] \approx \gamma'_{00}\, S_{(\mu\nu\sigma)} a^{+\mu}a^{+\nu}a^{+\sigma}\,. \tag{15}
$$

We see that to close the algebra we must demand $\gamma_{00} = \gamma'_{00} = 0$. This means that the totally symmetric tensor $S_{(\mu\nu\sigma)}$ cannot influence on the background geometry.

Next we consider commutators

$$
\begin{aligned}
[\,l_1, l_2^+] \;&\approx\; a^{+\mu}D_\mu + \alpha_{00}\, V_\mu a^{+\mu} - \beta_{00}\, M_{[\mu\nu]\sigma} a^{+\nu}a^{+\sigma}a^{\mu} \\
&\approx\; -l_1^+ + (\alpha_{00} + \alpha'_{00})\, V_\mu a^{+\mu} + (\beta'_{00} - \beta_{00})\, M_{[\mu\nu]\sigma} a^{+\nu}a^{+\sigma}a^{\mu}\,,
\end{aligned}
\tag{16}
$$

$$
[\,l_1^+, l_2] \;\approx\; l_1 - (\alpha_{00} + \alpha'_{00})\, V_\mu a^{\mu} + (\beta'_{00} - \beta_{00})\, M_{[\mu\nu]\sigma} a^{+\mu}a^{\nu}a^{\sigma}\,, \tag{17}
$$

and for their closing it is necessary to put $\alpha'_{00} = -\alpha_{00}$ and $\beta'_{00} = \beta_{00}$. Thus operators l_1 and l_1^+ will take the form

$$
l_1 \;\approx\; a^{\mu}D_\mu + \alpha_{00} V_\mu a^{\mu} + \beta_{00} M_{[\mu\nu]\sigma} a^{+\mu}a^{\nu}a^{\sigma} \tag{18}
$$

$$
l_1^+ \;\approx\; -a^{+\mu}D_\mu - \alpha_{00} V_\mu a^{+\mu} + \beta_{00} M_{[\mu\nu]\sigma} a^{+\nu}a^{+\sigma}a^{\mu} \tag{19}
$$

Let us now consider commutator

$$[l_1^+, l_1] \quad \sim \quad D^2 + P^{\mu\alpha\sigma} a_\mu^+ a_\alpha D_\sigma + 2\alpha_{00} V^\sigma D_\sigma + W^{\mu\nu\alpha\beta} a_\mu^+ a_\nu^+ a_\alpha a_\beta + K^{\mu\alpha} a_\mu^+ a_\alpha + Z\,, \qquad (20)$$

where

$$D^2 = g^{\mu\nu}(D_\mu D_\nu - \Gamma^\sigma{}_{\mu\nu} D_\sigma)\,, \qquad (21)$$

$$P^{\mu\alpha\sigma} = 2\beta_{00}(M^{\mu(\alpha\sigma)} - M^{\alpha(\mu\sigma)}) = -P^{\alpha\mu\sigma}\,, \qquad (22)$$

$$\begin{aligned} W_{\mu\nu\alpha\beta} &= R_{\mu\alpha\beta\nu} - \beta_{00}\big[\nabla_{(\beta} M_{\alpha)\mu\nu} + \nabla_{(\mu} M_{\nu)\alpha\beta}\big] \\ &\quad + \beta_{00}^2\big[M_{\tau(\mu\nu)} M^\tau{}_{(\alpha\beta)} - 4 M_{\alpha(\tau\mu)} M_{\nu(\beta}{}^{\tau)}\big]\,, \end{aligned} \qquad (23)$$

$$\begin{aligned} K_{\mu\alpha} &= R_{\mu\alpha} - 2\beta_{00}\nabla^\sigma M_{\alpha(\mu\sigma)} - 2\beta_{00}^2 M_{\mu(\rho\tau)} M_\alpha{}^{\rho\tau} \\ &\quad + \alpha_{00}(\nabla_\alpha V_\mu - \nabla_\mu V_\alpha) + 2\alpha_{00}\beta_{00}(M_{\mu(\alpha\sigma)} - M_{\alpha(\mu\sigma)})V^\sigma\,, \end{aligned} \qquad (24)$$

$$Z = \alpha_{00}^2 V_\mu V^\mu + \alpha_{00}\nabla_\mu V^\mu \qquad (25)$$

and $\sim$ means "up to terms proportional to operators l_1, l_1^+, l_2, l_2^+, g_0". In order to have a closed algebra we have to suppose that the right hand side of (20) be proportional to operators of the algebra (8). For example we may define operators l_0 and g_m as follows

$$l_0 \sim D^2 - m^2 + P^{\mu\alpha\sigma} a_\mu^+ a_\alpha D_\sigma + 2\alpha_{00} V^\sigma D_\sigma + W^{\mu\nu\alpha\beta} a_\mu^+ a_\nu^+ a_\alpha a_\beta + K^{\mu\alpha} a_\mu^+ a_\alpha + Z\,, \qquad (26)$$

$$g_m = m^2\,. \qquad (27)$$

Let us turn to the commutator $[l_1^+, l_0]$ and consider terms with two derivative operators. One has

$$[l_1^+, l_0] \quad \sim \quad P^{\alpha(\mu\sigma)} a_\alpha^+ D_{(\mu} D_{\sigma)} + \ldots \qquad (28)$$

Demanding that this commutator be proportional to operators (8) and since we are working with traceless fields, then one has to suppose

$$P^{\alpha(\mu\sigma)} = 0. \qquad (29)$$

If $\beta_{00} \neq 0$, then taking into account first (22) and then (9) we come to the conclusion that $M_{\mu\nu\sigma} = M_{[\mu\nu\sigma]} = 0$. To avoid $M_{\mu\nu\sigma} = 0$ we will adopt less strong condition that $\beta_{00} = 0$. In any case this means that the background field $M_{[\mu\nu]\sigma}$ cannot influence on the background geometry.

Now commutator $[l_1^+, l_0]$ takes the form

$$\begin{aligned} [l_1^+, l_0] \quad \sim \quad & 4a^{+\mu} a^{+\nu} a^\alpha R^\sigma{}_{\mu\nu\alpha} D_\sigma + (2R^{\mu\sigma} + 3\alpha_{00} F^{\sigma\mu}) a_\mu^+ D_\sigma - R_{\mu\alpha\beta\nu;\sigma} a^{\mu+} a^{\nu+} a^{\sigma+} a^\alpha a^\beta \\ & + (4\alpha_{00} R^\sigma{}_{\mu\nu\alpha} V_\sigma - R_{\mu\nu;\alpha} + \alpha_{00}\nabla_\mu F_{\nu\alpha}) a^{+\mu} a^{\nu+} a^\alpha \\ & + a^{+\mu}\alpha_{00}\Big[3\alpha_{00} F_{\sigma\mu} V^\sigma + \nabla^\nu F_{\nu\mu} + 2V^\sigma R_{\sigma\mu}\Big]\,, \end{aligned} \qquad (30)$$

where $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$. We see that commutator (30) does not proportional to the operators (8) if the curvature and the background vector field V_μ are arbitrary. To find conditions on the curvature and V_μ which are necessary for closing the algebra we decompose the Riemann tensor into irreducible parts

$$\begin{aligned} R_{\mu\nu\alpha\beta} \quad = \quad & C_{\mu\nu\alpha\beta} + \frac{1}{d-2}\left(\tilde{R}_{\mu\alpha} g_{\nu\beta} + \tilde{R}_{\nu\beta} g_{\mu\alpha} - \tilde{R}_{\mu\beta} g_{\nu\alpha} - \tilde{R}_{\nu\alpha} g_{\mu\beta}\right) \\ & + \frac{R}{d(d-1)}(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}), \end{aligned} \qquad (31)$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R$ is the traceless part of the Ricci tensor, and substitute this decomposition into (30)

$$\begin{aligned}
[l_1^+, l_0] \;\; \sim \;\; & 4a^{+\mu} a^{+\nu} a^{\alpha} C^{\sigma}{}_{\mu\nu\alpha} D_\sigma + 3\alpha_{00} F^{\sigma\mu} a^+_\mu D_\sigma - C_{\mu\alpha\beta\nu;\sigma} a^{\mu+} a^{\nu+} a^{\sigma+} a^{\alpha} a^{\beta} \\
& + \Big[\alpha_{00} 4 C^{\sigma}{}_{\mu\nu\alpha} V_\sigma + \tilde{R}_{\mu\alpha;\nu} - \tilde{R}_{\mu\nu;\alpha} + \alpha_{00} \nabla_\mu F_{\nu\alpha}\Big] a^{+\mu} a^{\nu+} a^{\alpha} \\
& + a^{+\mu} \Big[3\alpha_{00}^2 F_{\sigma\mu} V^\sigma + \alpha_{00} \nabla^\sigma F_{\sigma\mu} - \frac{(d-2)(d-4)}{4d(d-1)} \nabla_\mu R\Big].
\end{aligned} \tag{32}$$

From first term of r.h.s. of (32) we find that it is necessary to suppose

$$C_{\mu(\alpha\beta)\nu} = 0\,, \qquad\qquad \alpha_{00} F_{\mu\nu} = \alpha_{00}(\nabla_\mu V_\nu - \nabla_\nu V_\mu) = 0\,. \tag{33}$$

The left condition in (33) together with the index symmetry of the Weyl tensor tells us that the Weyl tensor is completely antisymmetric $C_{\mu\alpha\beta\nu} = C_{[\mu\alpha\beta\nu]}$, and due to the Bianchi identity $C_{\mu[\alpha\beta\nu]} = 0$ it equals to zero. To satisfy the right condition in (33) we can put $\alpha_{00} = 0$ or $F_{\mu\nu} = 0$, after that field V_μ disappears in the r.h.s. (32). In particular, if the coefficient α_{00} is somehow fixed, e.g. the vector field enters the constraints through the $U(1)$ covariant derivative, ones get immediately that $F_{\mu\nu} = 0$. This means in general that under the above assumptions the vector field V_μ does not influence on closing the algebra and as a consequence on the background geometry.

Next, from the second line of (32) we find

$$\tilde{R}_{\alpha(\mu;\nu)} = \tilde{R}_{\mu\nu;\alpha} \tag{34}$$

where we have used (33). Contracting indices μ and ν in (34) and using the Bianchi identity one gets

$$\nabla_\mu R = 0 \quad \Rightarrow \quad R = const. \tag{35}$$

As a result, the background geometry must be a constant curvature space-time.

Thus we have shown that in the higher spin field theory the background vector and third rank tensor cannot have influence on the geometry of the space which must be only a constant curvature one.

Let us consider the case when the background fields are introduced into the operators being multiplied on some operators of the algebra, like, for example, they are introduced in the operator l_1 (11), except the terms with coefficients α_{00}, β_{00}, γ_{00}. In this case we expect that the background fields will make no effects, at least on the physical field, the same as in quantization of gauge theories a redefinition of constraints by terms proportional to constraints have no effect on the physical states. In case of higher spin theory we illustrate this on a simplified example.

Let the operators have the following form

$$l_0 = \partial^2 - m^2 + \alpha_1 g_0 + \alpha_2 g_0^2 \tag{36}$$

$$l_1 = a^\alpha \partial_\alpha \qquad\qquad l_1^+ = -a^{+\mu}\partial_\mu \qquad\qquad g_m = m^2 \tag{37}$$

$$l_2 = \tfrac{1}{2} a_\mu a^\mu \qquad\qquad l_2^+ = \tfrac{1}{2} a^+_\mu a^{+\mu} \qquad\qquad g_0 = a^+_\mu a^\mu + \tfrac{d}{2} \tag{38}$$

where α_1 and α_2 are some combinations of the background fields with the dimension of mass squared. Since α_1 and α_2 are multiplied on operator of the algebra, then the algebra is closed

at any α_1 and α_2. To simplify the subsequent calculations we adopt α_1 and α_2 are constants. At first glance it seems that the mass shell equation which we will reproduce using the BRST method must will be $l_0|\Phi\rangle = 0$, but as we shall show it will turn out to be $(\partial^2 - m^2)|\Phi\rangle = 0$, thus removing all the dependence on α's (this is what we mean in footnote 2 at page 4). In case of constant α's the algebra of operators (36)–(38) has the following non-vanishing commutators

$$
\begin{aligned}
&[l_1^+, l_1] = l_0 + g_m - \alpha_1 g_0 - \alpha_2 g_0^2, && && (39)\\
&[l_0, l_1] = -2\alpha_2 g_0 l_1 - (\alpha_1 + \alpha_2) l_1, && [l_2^+, l_1] = l_1^+, && (40)\\
&[l_0, l_1^+] = 2\alpha_2 l_1^+ g_0 + (\alpha_1 + \alpha_2) l_1^+, && [l_1^+, l_2] = l_1, && (41)\\
&[l_0, l_2] = -4\alpha_2 g_0 l_2 - (2\alpha_1 + 4\alpha_2) l_2, && [l_2, l_2^+] = g_0, && (42)\\
&[l_0, l_2^+] = 4\alpha_2 l_2^+ g_0 + (2\alpha_1 + 4\alpha_2) l_2^+, && [g_0, l_k^\pm] = \pm k l_k. && (43)
\end{aligned}
$$

According to the BRST method of Lagrangian construction (see e.g. [8]) since among the operators (36)–(38) there are operators g_0, g_m which are not constrains neither in the space of bra vectors nor in the space of ket vectors then we must construct extended expressions for the operators $o_i \to O_i = o_i + o'_i$ where o'_i are additional parts to the initial operators $o_i = \{l_0, l_1, l_1^+ l_2, l_2^+, g_0, g_m\}$ (36)–(38). These additional parts are constructed from new (additional) creation and annihilation operators and commute with the initial operators $[o_i, o'_j] = 0$. The extended expressions for the operators must satisfy two conditions: 1) they must form an algebra $[O_i, O_j] \sim O_k$; 2) the operators which are not constraints g_0, g_m must be zero or contain linearly arbitrary parameters which value will be defined later from the condition of reproducing desired equations of motion.

Using the method elaborated in [8] we find algebras of the additional parts

$$
\begin{aligned}
&[l'_1, l'^+_1] = -l'_0 - g'_m + \alpha_1 g'_0 - \alpha_2 g'^2_0, && && (44)\\
&[l'_1, l'_0] = -2\alpha_2 g'_0 l'_1 + (\alpha_1 - \alpha_2) l'_1, && [l'^+_2, l'_1] = l'^+_1, && (45)\\
&[l'_0, l_1^{+\prime}] = -2\alpha_2 l_1^{+\prime} g'_0 + (\alpha_1 - \alpha_2) l_1^{+\prime}, && [l'_1+, l'_2] = l'_1, && (46)\\
&[l'_2, l'_0] = -4\alpha_2 g'_0 l'_2 + (2\alpha_1 - 4\alpha_2) l'_2, && [l'_2, l'^+_2] = g'_0, && (47)\\
&[l'_0, l_2^{+\prime}] = -4\alpha_2 l_2^{+\prime} g'_0 + (2\alpha_1 - 4\alpha_2) l_2^{+\prime}, && [g'_0, l'^\pm_k] = \pm k l'^\pm_k && (48)
\end{aligned}
$$

and of the extended operators

$$
\begin{aligned}
&[L_1^+, L_1] = L_0 + G_m - \alpha_1 G_0 - \alpha_2 G_0^2 + 2\alpha_2 g'_0 G_0 && && (49)\\
&[L_1, L_0] = \alpha_2(G_0 L_1 + L_1 G_0) + \alpha_1 L_1 - 2\alpha_2 g'_0 L_1 - 2\alpha_2 l'_1 G_0, && [L_2^+, L_1] = L_1^+, && (50)\\
&[L_0, L_1^+] = \alpha_2(L_1^+ G_0 + G_0 L_1^+) + \alpha_1 L_1^+ - 2\alpha_2 g'_0 L_1^+ - 2\alpha_2 l'^+_1 G_0, && [L_1^+, L_2] = L_1, && (51)\\
&[L_2, L_0] = 2\alpha_2(G_0 L_2 + L_2 G_0) + 2\alpha_1 L_2 - 4\alpha_2 g'_0 L_2 - 4\alpha_2 l'_2 G_0, && [L_2, L_2^+] = G_0, && (52)\\
&[L_0, L_2^+] = 2(L_2^+ G_0 + G_0 L_1^+) + 2\alpha_1 L_2^+ - 4\alpha_2 g'_0 L_2^+ - 4\alpha_2 l_2^{+\prime} G_0, && [G_0, L_k^\pm] = \pm k L_k^\pm. && (53)
\end{aligned}
$$

In RHS of (49)–(53) we choose symmetric odering of the extended operators. There is a method allowing to construct explicit form of operators in terms of creation and annihilation operators on the base of their algebra, see e.g. [8] and references therein. But for our purpose we need no any explicit realization of the additional parts, except only one observation

$$
l'_0 = \alpha_1 g'_0 - \alpha_2 g'^2_0. \tag{54}
$$

The BRST operator constructed on the base of the algebra of the extended operators is

$$\begin{aligned}
\tilde{Q} &= \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_G G_0 + \eta_M G_m + \eta_1^+ \eta_1 (\mathcal{P}_0 + \mathcal{P}_M) \\
&\quad + (\eta_G \eta_1^+ + \eta_2^+ \eta_1)\mathcal{P}_1 + (\eta_1 \eta_G + \eta_1^+ \eta_2)\mathcal{P}_1^+ + 2\eta_G \eta_2^+ \mathcal{P}_2 + 2\eta_2 \eta_G \mathcal{P}_2^+ \\
&\quad - \eta_2^+ \eta_2 \mathcal{P}_G - \eta_1^+ \eta_1 \Big[\alpha_1 + \alpha_2 (G_0 - 2g_0')\Big]\mathcal{P}_G \\
&\quad + \alpha_2 \eta_0 \Big[\eta_1^+ (L_1 - 2l_1') - \eta_1 (L_1^+ - 2l_1^{+\prime}) + 2\eta_2^+ (L_2 - 2l_2') - 2\eta_2 (L_2^+ - 2l_2^{+\prime})\Big]\mathcal{P}_G \\
&\quad + \eta_0 \Big[\alpha_1 + \alpha_2 (G_0 - 2g_0')\Big]\Big(\eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ + 2\eta_2^+ \mathcal{P}_2 - 2\eta_2 \mathcal{P}_2^+\Big),
\end{aligned} \tag{55}$$

Next step of Lagrangian construction is determination of the arbitrary parameters which must be contained linearly in the additional parts g_0' and g_m'. For this we decompose the BRST operator extracting its dependence on ghosts η_G, $\mathcal{P}_G$, η_M, $\mathcal{P}_M$, corresponding to these operators,

$$\tilde{Q} = Q + \eta_G \tilde{G}_0 + \eta_M G_M + \eta_1^+ \eta_1 \mathcal{P}_M + \mathcal{B}\mathcal{P}_G \tag{56}$$

where

$$\begin{aligned}
Q &= \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_1^+ \eta_1 \mathcal{P}_0 + \eta_2^+ \eta_1 \mathcal{P}_1 + \eta_1^+ \eta_2 \mathcal{P}_1^+ \\
&\quad + \eta_0 \Big[\alpha_1 + \alpha_2 (G_0 - 2g_0')\Big]\Big(\eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ + 2\eta_2^+ \mathcal{P}_2 - 2\eta_2 \mathcal{P}_2^+\Big),
\end{aligned} \tag{57}$$

$$\tilde{G}_0 = G_0 + \eta_1^+ \mathcal{P}_1 - \eta_1 \mathcal{P}_1^+ + 2\eta_2^+ \mathcal{P}_2 - 2\eta_2 \mathcal{P}_2^+, \tag{58}$$

(explicit expression for $\mathcal{B}$ is not essential) and suppose that the state vector $|\Psi\rangle$ in the extended space including ghosts does not depend on ghosts η_G and η_M, $\mathcal{P}_G|\Psi\rangle = \mathcal{P}_M|\Psi\rangle = 0$. As a result the equation defining physical states $\tilde{Q}|\Psi\rangle = 0$ is decomposed into three equations

$$Q|\Psi\rangle = 0, \qquad \tilde{G}_0|\Psi\rangle = 0, \qquad G_M|\Psi\rangle = 0. \tag{59}$$

Two right equations in (59) are used for determination of the arbitrary constants in g_0' and g_m' and the left equation in (59) is the equation on physical states. Note that using (54) and $L_0 = l_0 + l_0'$ where l_0 is given by (36) operator Q (57) can be rewritten as

$$\begin{aligned}
Q &= \eta_0 \Big[\partial^2 - m^2 + \alpha_1 \tilde{G}_0 + \alpha_2 (G_0 - 2g_0')\tilde{G}_0\Big] \\
&\quad + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_1^+ \eta_1 \mathcal{P}_0 + \eta_2^+ \eta_1 \mathcal{P}_1 + \eta_1^+ \eta_2 \mathcal{P}_1^+
\end{aligned} \tag{60}$$

and due to the middle equation in (59) all the effects of the "constant background fields" α_1 and α_2 disappear and we get the model of free higher spin field in Minkowski space.

To summarize, we have developed the BRST approach to Lagrangian construction for bosonic totally symmetric higher spin field in external gravitational, vector and third rank tensor fields. Assuming that interaction with external fields has massless and flat space limits we prove that the consistent formulation is possible only in constant curvature space with no nontrivial coupling to the third rank tensor and the strength of the vector fields. One can expect that analogous situation will take place for higher spin fermionic fields and for any deformation of constant curvature space by more general background tensor fields. However, the above result does not concern the field models with spins $\frac{3}{2}$ and 2 where the BRST construction has the specific possibilities [9] and allows the consistent Lagrangian formulation in Einstein

spaces. Thus, the further development of Lagrangian construction for free higher spin fields interacting with external fields is related to search for interaction Lagrangians which contain the inverse mass terms. Some approaches to such Lagrangians are given in refs. [3, 4]. Also, it would be interesting to study the consistency conditions for for recently formulated conformal higher spin fields [16] if to couple them to external fields.

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